Regularized Iterative Weighted Filtered Backprojection for Helical Cone-Beam CT

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Abstract—An iterative filtered backprojection method for helical cone-beam CT has been examined. This method is based on iterative application of a non-exact 3DFBP method able to handle redundant data at arbitrary pitches without significant detector masking. In contrast to statistical methods, this method is expected to deliver a good result in less than five iterations.

Artifact reduction, as well as spatial resolution and noise properties, have been measured. Even without regularization, during the first five iterations, cone artifacts are clearly reduced. As a side effect, however, the spatial resolution and noise are increased. To avoid this side effect and improve the convergence properties, a regularization proposed and evaluated.

I. INTRODUCTION

This contribution is an extension of theory and experiments on iterative weighted filterd backprojection (IWFBP) presented in [9]. The main difference from previous work is that more detailed and correct noise and spatial resolution investigations have been made, and the addition of a regularization operation that improve the convergence properties of the iterative loop. In the present investigation we abandoned the ordered subset approach in [9] because it amplifies noise and it is not clear how to combine it with the regularization presented here.

We have applied the iterative improvement scheme illustrated in Fig. 1 to the non-exact weighted filtered backprojection (WFBP) method proposed by Stierstorfer et al. [8]. This iterative scheme is a special case of the iterative filtered backprojection (IFBP) methods analyzed by Xu et al. [11]. IFBP methods have successfully been used for attenuation correction in single photon emission computed tomography (SPECT) [1], [10], reduction of streaks due to missing angles [6], [2], and for reduction of artifacts due to an incomplete focus trajectory in cone-beam CT [14].

A problem with exact analytical methods presented for helical cone-beam CT seems to be that they are unable to utilize redundant data for arbitrary table feeds [8]. The IWFBP method presented here does not suffer from this drawback, and is significantly faster than statistical methods, which has motivated the experiments presented here.

II. METHODS

A. Weighted filtered backprojection (WFBP)

A detailed description of the WFBP method is found in the paper by Stierstorfer et al. [8]. We repeat the main steps here.

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Fig. 1: Illustration of the iterative weighted filtered backprojection (IWFBP) method. First, input data are rebinned to semi-parallel geometry. Given an initial image vector \( f_0 \in \mathbb{R}^N \), a sequence of image vectors is generated by the update formula \( f_{k+1} = f_k + \alpha Q(p_{reb} - Pf_k) \). The matrices \( Q \in \mathbb{R}^{N \times M} \) and \( P \in \mathbb{R}^{M \times N} \) correspond to the WFBP method and a projection operator modeling the acquisition process respectively.

1. Rebinning to semi-parallel geometry. The word “semi” is used here because the projections of rays onto the axial plane are parallel, while the rays diverge in the z-direction.
2. Down-weighting of rows located close to the borders of the detector. A parameter \( Q \in (0, 1) \) is used for controlling the amount of down-weighting.
3. Row-wise rampfiltering of projection data.

In the following, the matrix \( Q \) will be used for representing step (2) to (4) above.

B. Iterative weighted filtered backprojection (IWFBP)

Let \( N \) be the number of voxels and \( M \) be the total number of x-ray attenuation measurements. Furthermore, let \( p_{in} \in \mathbb{R}^M \) denote input data and \( f_0 \in \mathbb{R}^N \) denote a vector representing an initial voxel volume. The update step of IWFBP is then given by

\[
f_{k+1} = f_k + \alpha Q(p_{reb} - Pf_k)
\]

where \( Q \) is the reconstruction matrix from the previous section and \( P \in \mathbb{R}^{M \times N} \) is a projection matrix. In this way, a sequence of voxel volumes \( \{f_0, f_1, \ldots \} \) is produced. Our experiments show that only a few iterations are needed to obtain a significant reduction of cone artifacts.

C. Regularization and convergence

The following presentation on quadratic regularization for least square methods is inspired by the text on regularization by De Man [5]. We believe quadratic regularization is of
special interest since it leads to a linear contribution in the update step. Suppose that we want to find the voxel volume \( f \) that minimizes

\[
z(f) = \frac{1}{2} \| Pf - p_n \|^2 + \beta \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij}(f_i - f_j)^2 \tag{2}
\]

where \( d_{ij} \) are the inverse distances between the voxels \( i \) and \( j \) in a 3\textsuperscript{rd} neighborhood, and \( \beta \) is a parameter determining the amount of regularization. The last term is obviously the regularization part. Differentiation of (2) yields

\[
\nabla z(f) = Pf - p_n + 4\beta \sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij}(f_i - f_j) e_i \tag{3}
\]

where \( \{e_1, e_2, ..., e_N\} \) is the standard basis for \( \mathbb{R}^N \). Thus, a steepest descent method for minimizing \( z(f) \) is given by

\[
f_{k+1} = f_k - \alpha P^T(Pf_k - p_n) - 4\alpha \beta Rf_k. \tag{4}
\]

In IWFBP, the regularized version of IWFBP, the last term in (4) has been added to the IWFBP update step (without the factor 4), resulting in

\[
f_{k+1} = f_k + \alpha Q(p_{\text{reb}} - Pf_k) - \alpha \beta Rf_k. \tag{5}
\]

Zeng and Gullberg [13] showed that for non-regularized IFBP methods, the sequence of voxel volumes converges to

\[
f_\infty = (QP)^{-1}Qp_n \tag{6}
\]

if all eigenvalues of \( \alpha QP \) are contained in the interior of a unit disc centered around 1. The corresponding criterion for convergence of the regularized IWFBP is that all eigenvalues of \( \alpha (QP + \beta R) \) should be contained in the same unit disc.

In order to suppress edge and aliasing artifacts, Zbijewski and Beekman [12] suggested the use of a densely sampled voxel volume. With such oversampling, the non-regularized IWFBP obviously fails to satisfy the convergence criterion since \( N > M \) implies a non empty null space \( N(P) \). However, we feel there is a strong conjecture that regularization eliminates this problem. We may safely assume that the nullspace \( N(P) \) consists of high-frequency structures. Since the matrix \( R \) corresponds to the high-pass filter illustrated in Fig. 2, such structures do not belong to the nullspace of \( (QP + \beta R) \).

III. RESULTS

A. Experimental setup

The Thorax phantom by Sourbelle [7] was used for visual inspection and calculation of the error measure

\[
\sigma_e = \frac{1}{|\Omega_e|} \sum_{i \in \Omega_e} ((f_{\text{noise-free}})_i - (f_{\text{phant}})_i)^2 \tag{7}
\]

and noise measure

\[
\sigma_n = \frac{1}{|\Omega_n|} \sum_{i \in \Omega_n} ((f_{\text{noise}})_i - (f_{\text{noise-free}})_i)^2. \tag{8}
\]

Here, \( f_{\text{noise}} \) and \( f_{\text{noise-free}} \) represent reconstructions from noisy and noise-free projections respectively. The vector \( f_{\text{phant}} \) corresponds to a sampled phantom and \( \Omega_e \) and \( \Omega_n \) are the sets of voxels, which over errors and noise are measured. These sets are indicated in Fig. 3.

Noise-free projection data for all experiments were generated by the simulation software CTSIM (Siemens Medical, Forchheim). Noise were added as suggested by Fuchs [3]. For each detector measurement, 3\textsuperscript{rd} rays were used for modeling finite focus, finite detector, and gantry rotation. Other parameters for scanning and reconstruction are shown in Table I.

For evaluating spatial resolution, modulation transfer functions (MTFs) and slice sensitivity profiles (SSPs) have been measured. The MTFs were measured using the edge method described by Judy [4]. SSPs were measured by reconstructing a phantom consisting of several very thin discs with a known displacement in the z-direction. By assuming that the SSPs are invariant to position in the z-direction, supersampled SSPs have been measured by merging measurements from several discs.

B. Cone artifact reduction and convergence

Fig. 4 shows result images from the first two iterations. It is clear from these images that the cone artifacts present in \( f_1 \) to a large extent are suppressed during these two iterations.
As the number of iterations increases for the non-regularized IWFBP, we observe not only suppression of cone artifacts, but also an increase of overshoots, spatial resolution, and noise. By choosing an appropriate value for the regularization parameter $\beta$, the frequency and noise characteristics of the original WFBP method can be approximately preserved. However, even if the overshoot from the vertebra is reduced by the regularization, a significant part thereof remains visible.

The quantity $\|f_{k+1} - f_k\|$ can be used as an indicator of convergence (or divergence) since the sequence $\{f_0, f_1, \ldots\}$ converges only if $\|f_{k+1} - f_k\|$. Fig. 5 shows this norm and the $\sigma_e$ values plotted against the number of iterations. For the non-regularized IWFBP, the norm is reduced to about 1/30 in the first five iterations. However, even after 40 iterations the norm does not fall below 1/50 of the first iteration. Therefore, it is difficult to tell anything about the convergence in the non-regularized case. In contrast, the corresponding norm for the regularized IWFBP with $\beta = 0.0005$ falls to 1/50 in the first five iterations and continues to drop down to 1/100000 after 30 iterations. In terms of $\sigma_e$ values (see Eq. (7) and Fig. 3 for definitions), the non-regularized IWFBP is clearly outperformed by the regularized IWFBP.

**C. Spatial resolution and noise**

MTFs for different values of $\beta$ are shown in Fig. 6. Compared to the WFBP, clearly $\beta = 0.0000$ and $\beta = 0.0003$ result in higher spatial resolution in the $xy$-plane, while $\beta = 0.0007$ results in a slightly lower resolution. For $\beta = 0.0005$, the MTF of IWFBP after 6 iterations is approximately equal to the MTF of WFBP.

Fig. 7 shows the lower part of the normalized SSPs, and Table II shows full widths at half maximum (FWHMs) for different values of $\beta$. Examining the FWHMs, the IWFBP with $\beta = 0.0007$ comes closest to the WFBP method. However, in terms of low overshoots in the $z$-direction, none of the IWFBP reconstructions match the original WFBP reconstruction very well. For $\beta = 0.0005$, which resulted in an MTF similar to the WFBP, the frequency and noise characteristics of the original WFBP method can be approximately preserved.

As the number of iterations increases for the non-regularized IWFBP, the norm is reduced to about 1/30 in

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**Table I:** Scanning and reconstruction parameters

<table>
<thead>
<tr>
<th>Scanning parameters</th>
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<tbody>
<tr>
<td>Source-isocenter distance</td>
<td>570 mm</td>
<td></td>
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<tr>
<td>Number of channels (Quarter offset)</td>
<td>336</td>
<td></td>
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<tr>
<td>Number of rows</td>
<td>64</td>
<td></td>
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<tr>
<td>Number of projections per turn</td>
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<td></td>
</tr>
<tr>
<td>Focus width ($xy$)</td>
<td>1.3 mm</td>
<td></td>
</tr>
<tr>
<td>Effective focus length ($z$)</td>
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<td></td>
</tr>
<tr>
<td>Slice width</td>
<td>1.5 mm</td>
<td></td>
</tr>
<tr>
<td>Detector height</td>
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<td></td>
</tr>
<tr>
<td>Table feed</td>
<td>96 mm</td>
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<tr>
<td>Maximal fan angle</td>
<td>$\pm 26^\circ$</td>
<td></td>
</tr>
<tr>
<td>Maximal cone angle</td>
<td>$\pm 4.8^\circ$</td>
<td></td>
</tr>
<tr>
<td>Number of photons per measurement</td>
<td>$10^9$</td>
<td></td>
</tr>
</tbody>
</table>

**Detector sampling after rebinning**

| Number of channels | 672 | |
| Maximum parallel displacement | $\pm 259$ mm | |

**Reconstruction parameters**

| Number of voxels | $513 \times 513 \times 257$ | |
| Width of reconstructed volume | 500 mm | |
| Height of reconstructed volume | 192.75 mm | |

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**Fig. 4:** Non-regularized and regularized ($\beta = 0.0005$) IWFBP reconstructions. The slice shown here is located very close to the vertebra, hence the strong initial cone artifact. Greyscale window $\pm 30$ HU.

**Fig. 5:** To the left, $\log_{10}(|f_{k+1} - f_k|/C)$ is plotted as a function of the number of iterations. The normalization parameter $C$ has been chosen so that the largest value equals 1. To the right, the error $\sigma_e$ is plotted against the number of iterations.

**Fig. 6:** Modulation transfer functions (MTFs) for different values of the regularization parameter $\beta$. The MTFs for different values of $\beta$ are shown in Fig. 6. Compared to the WFBP, clearly $\beta = 0.0000$ and $\beta = 0.0003$ result in higher spatial resolution in the $xy$-plane, while $\beta = 0.0007$ results in a slightly lower resolution. For $\beta = 0.0005$, the MTF of IWFBP after 6 iterations is approximately equal to the MTF of WFBP.

**Fig. 7:** The lower part of the normalized SSPs, and Table II shows full widths at half maximum (FWHMs) for different values of $\beta$. Examining the FWHMs, the IWFBP with $\beta = 0.0007$ comes closest to the WFBP method. However, in terms of low overshoots in the $z$-direction, none of the IWFBP reconstructions match the original WFBP reconstruction very well. For $\beta = 0.0005$, which resulted in an MTF similar to the WFBP, the frequency and noise characteristics of the original WFBP method can be approximately preserved.
Fig. 7: Slice sensitivity profiles (SSPs) for different values of the regularization parameter $\beta$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\sigma_e$ (HU)</th>
<th>$\sigma_n$ (HU)</th>
<th>FWHM (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WFBP</td>
<td>16.21</td>
<td>3.50</td>
<td>2.01</td>
</tr>
<tr>
<td>0.0000</td>
<td>4.00</td>
<td>5.78</td>
<td>1.62</td>
</tr>
<tr>
<td>0.0003</td>
<td>3.41</td>
<td>4.43</td>
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</tr>
<tr>
<td>0.0005</td>
<td>3.27</td>
<td>3.79</td>
<td>1.90</td>
</tr>
<tr>
<td>0.0007</td>
<td>3.21</td>
<td>3.28</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Table II: $\sigma_e$, $\sigma_n$ and full widths at half maximum (FWHM) of the SSPs for different values of $\beta$. The number of iterations is five.

Table II shows $\sigma_e$ and $\sigma_n$ values for different values of $\beta$. Clearly, both these are reduced as $\beta$ is increased. Again, the noise level for IWFBP is approximately the same as for WFBP when the value of $\beta$ is between 0.0005 and 0.0007.

IV. CONCLUSIONS

From the experiments presented here, we conclude that during the first five iterations, the IWFBP scheme efficiently suppresses cone artifacts produced by WFBP at a cone angle of $\pm 4.8^\circ$. The question regarding final convergence of the non-regularized IWFBP remains, but it has been shown that the behavior can be stabilized by introducing the regularization matrix $R$.

The frequency and noise characteristics of IWFBP obviously change as a function of the number of iterations. However, by choosing an appropriate value of the regularization parameter $\beta$, the characteristics become similar to those of the WFBP reconstruction. With regularization, the IWFBP reaches the final solution faster than without. It is therefore easier when to terminate the iterative loop.

REFERENCES


